

A New Way of Teaching the Special Theory of Relativity

by Michael Pohlig and Hans Michael Strauch

1 What changes Newtonian Mechanics into the Special Theory of Relativity?

Usually, in introducing in the Special Theory of Relativity we begin by discussing the failure of the MICHELSON-MORLEY experiment. FITZGERALD's hypothesis, elaborated by LORENTZ stated that all bodies in motion should be shortened in the direction of their velocity. He believed this contraction to be caused by special molecular forces. Very different to this explanation was the assumption made by EINSTEIN:

Axiom: The velocity of light in empty space is the same in all reference frames and is independent of the motion of the emitting body.

This axiom turns NEWTONIAN mechanics into EINSTEIN's Special Theory of Relativity. Using this axiom, EINSTEIN could prove the very important theorem:

*Theorem: The mass of a body and the energy of a body are just different words for the same physical quantity.*¹

The well-known formula for this theorem is:

$$E = m \cdot c^2 \quad (i)$$

In the Karlsruhe Physics Course² EINSTEIN's axiom and theorem change places. The sentence - *mass and energy are the same physical quantity* - is now the axiom which changes NEWTONIAN mechanics into EINSTEIN's Special Theory of Relativity and the sentence - *the velocity of light in empty space is the same in all reference frames and is independent of the motion of the emitting body* - becomes a theorem. For mathematicians such an exchange is not unusual. One reason for exchanging an axiom with a theorem is that it actually aids comprehension of the problem. This paper will show, that many important results from the Special Theory of Relativity can be developed from the axiom: *Mass and energy are the same physical quantity*.

¹ EINSTEIN writes in his article: 'Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?' : „Die Masse eines Körpers ist ein Maß für seinen Energieinhalt; ändert sich seine Energie L , so ändert sich die Masse in dem selben Sinne um $L/9 \cdot 10^{20}$, wenn die Energie in Erg und die Masse in Gramm gemessen wird.“ (quoted from [1]). Translation: The mass of a body is a measurement of its energy, if its energy L changes, so the mass will change in the same way by $L/9 \cdot 10^{20}$ (if the energy is measured in erg and the mass in gramme). In [2] EINSTEIN writes: „Nach der Relativitätstheorie gibt es keinen prinzipiellen Unterschied zwischen Masse und Energie,...“ and „Masse ist Energie.“ Translated: "According to the Special Theory of Relativity there is, principally, no difference between mass and energy,... " and "Mass is energy..."

² The Karlsruhe physics course was developed by F. HERRMANN, university Karlsruhe, Germany.
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Formula (i) is often misunderstood. People think the formula means that energy could be changed into mass or vice versa - mass could be changed into energy. Such an interpretation of the formula leads to error if it is used in the same physical system. It would mean that in a system energy could increase at the cost of mass and vice versa - the mass could increase at the cost of energy. This, however, is not correct. In fact, mass and energy are different words for the same physical quantity. The factor c^2 in formula (i) converts the unit Kilogram (kg) into Joule (J), nothing more. It would be better to write formula (i) as

$$E = k \cdot m \text{ (i')}$$

The fact that $c = \sqrt{k}$ is a velocity, with a certain role in Einstein's Special Theory of Relativity, will become evident later. In school formula (i') should be used; to save time and space formula (i) is used in this paper.

2 The role of momentum and energy in the description of bodies in motion

In Newtonian mechanics the primary concepts are trajectory, velocity, mass and force. Momentum and energy are nothing more than tools for easier calculations. In contrast to this, the Karlsruhe Physics Course is based on quantities which are primary quantities of quantum mechanics. These quantities have something in common, namely, each can be pictured as a kind of "stuff". This is the reason why they are called "substance-like" [3]. Like any substance these quantities can be thought to be brought into a physical system, they can flow from one system to another and there exists currents of such quantities. Momentum \mathbf{p} and energy E , respectively, are such "substance-like" quantities. Their currents are traditionally called force \mathbf{F} and power P . We prefer to use the names "momentum current" instead of force and "energy current" instead of power. The pictures these names create in our minds are more practical than the ones created by the traditional names.

Here is a short summary of the most important rules pupils should know when they follow the Karlsruhe Physics Course:

- (1) A body in motion has momentum. If the mass and the velocity of the body are known, we can calculate its momentum by the formula:

$$p = m \cdot v \text{ (ii)}^3$$

A body with the mass 1kg and the velocity 1m/s has a momentum of 1Hy⁴

- (2) Momentum can flow from one body to another, providing that there is a conducting connection between both bodies.

If in the time interval Δt the momentum Δp flows into a body, we say, the momentum current into the body is:

³ We discuss motions in only one direction, hence we consider only one component of the vector $\mathbf{p}=(p_x,p_y,p_z)$. If there is only the x-component, we omit the index 'x'. In this paper this is valid for all vectors.

⁴ Hy is the abbreviation for Huygens.

$$I_p = F = \frac{\Delta p}{\Delta t} \text{ (iii) }^5$$

The unit of the momentum current is 1Newton ($1\text{N} = 1 \frac{\text{Hy}}{\text{s}}$)

(3) If momentum carries energy, the energy current is given by:

$$P = v \cdot F \text{ (iv)}$$

3 Model-building and the Karlsruhe physics course [4]

In this paper the model-building software Powersim™ [5] is used. A great advantage of a model-building system like Powersim and others like Stella [6] is that models can be edited by using graphic symbols. Another advantage is that these flow diagram symbols, used by Powersim⁶, allow us to build a picture of the substance-like quantities in our minds. In the model (Fig. 1) a level symbol is shown. It represents the quantity X , the momentum p for example. An arrow leads from a cloud into the level symbol. If X is momentum p , then there is a current of momentum and momentum will be accumulated. The fact that the arrow starts in a cloud shows us that it is not specified where the momentum current comes from.

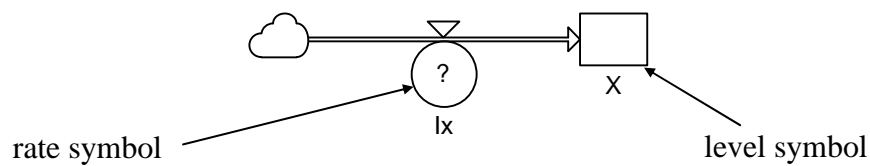


Figure 1: Some symbols of Powersim

The model shown in fig 1. represents the following iteration loop: the rate of change ΔX of X in a region is equal to the current into or out of the region, and the quantity X obeys a general conservation law.

$$X_{new} := X_{old} + \Delta X = X_{old} + I_X(t) \cdot \Delta t$$

The loop will be executed from t_1 to t_2 with a time step Δt . The initial value of the quantity X , that is X_{old} for the first run of the loop, and the time step Δt can be chosen by the user of the program. The iteration loop will be executed automatically. This is programmed by editing the model using the graphic symbols, as shown in fig 1. Another advantage of the model building program is that $I_X(t)$ must not be constant. When pupils first start to work with a model building system, very simple models should be chosen to familiarise pupils with such systems.

⁵ Currents are often designated by the symbol ' I_X ', while the index X indicates the quantity which is flowing. If X is the momentum p , traditionally the momentum current is named F . Each component of momentum obeys a general conservation law, hence the flow of momentum into a region is equal to the rate of momentum inside the region.

⁶ MIT - standard

3.1 Free falling of a body - the non relativistic view

One simple example is free falling of a body. Fig 2 shows a stone hanging on gallows. The spring is stretched. Thus momentum is leaving the stone through the spring. Since nothing is in motion, no momentum is accumulated. If momentum is leaving the stone but the stone remains in rest, a momentum current of the same amount must enter it. Its value is:

$$F = m \cdot g \quad (v)$$

Here g is the acceleration due to gravity. If we cut the connection between the stone and the spring, momentum cannot leave the stone anymore, thus the stone will fall to earth. Using the equations (ii), (iii), and (v) we find $v = g \cdot t$, the velocity increases linearly with time. This is shown by several well-known experiments.

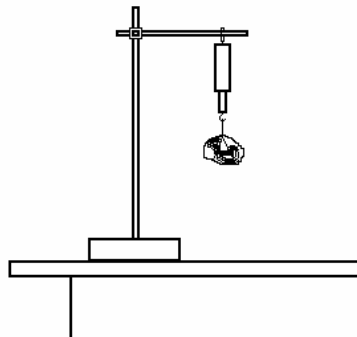


Figure 2: Momentum flows through the gravitation field into the stone and through the bars back to earth

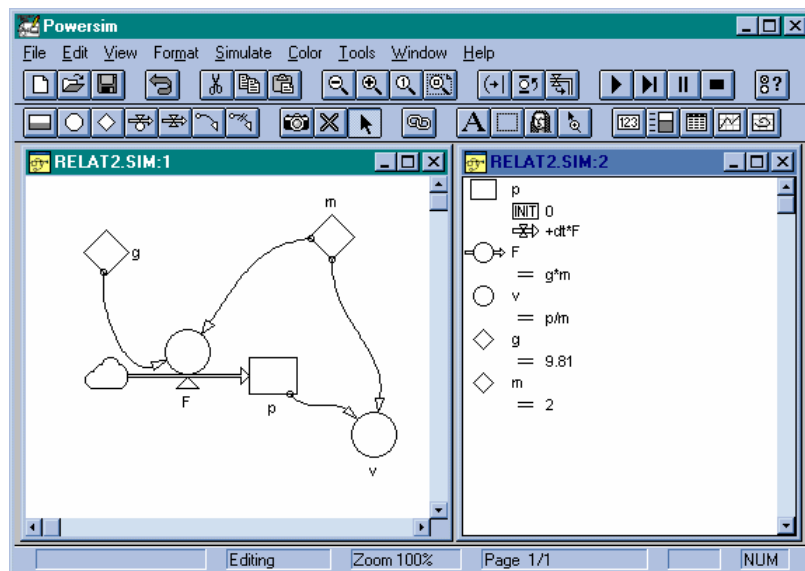


Figure 3: View of a model in a Powersim application window: diagram (left) equations (right)

Fig. 3 shows the Powersim application window with two views of the model 'free falling' in the workspace, each in separate smaller windows. The left one, the diagram view, displays the model's structure using the flow diagram symbols. The mass m and acceleration g are generated as constants, the velocity as a variable. Constants are displayed as rhombuses, variables as circles. The curved lines from the mass symbol to the velocity symbol and from the mo-

momentum symbol to the velocity symbol represent the links between these quantities. The velocity v is defined as:

$$v = \frac{p}{m}$$

The momentum must be initialized. Here the initial value is taken to be 0 Hy, this means that at the beginning of the simulation the body is not in motion. The momentum current is set to constant 10N and the mass to 2kg. Fig. 4 shows the simulation setup, that is: start time, stop time and time step. The iteration loop is calculated as a numerical integration. In the model we use the integration method RUNGE-KUTTA with variable step size⁷.

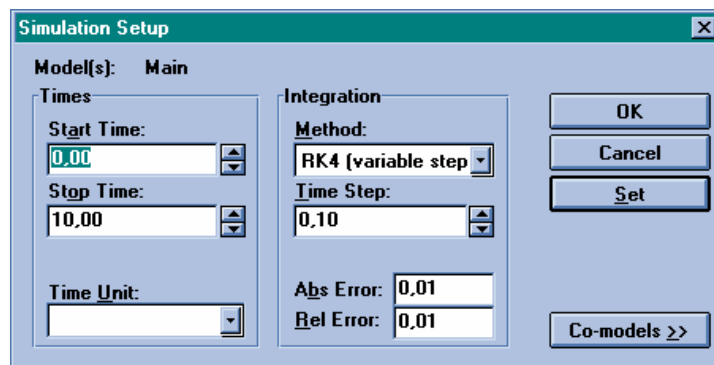


Figure 4: Simulation Setup

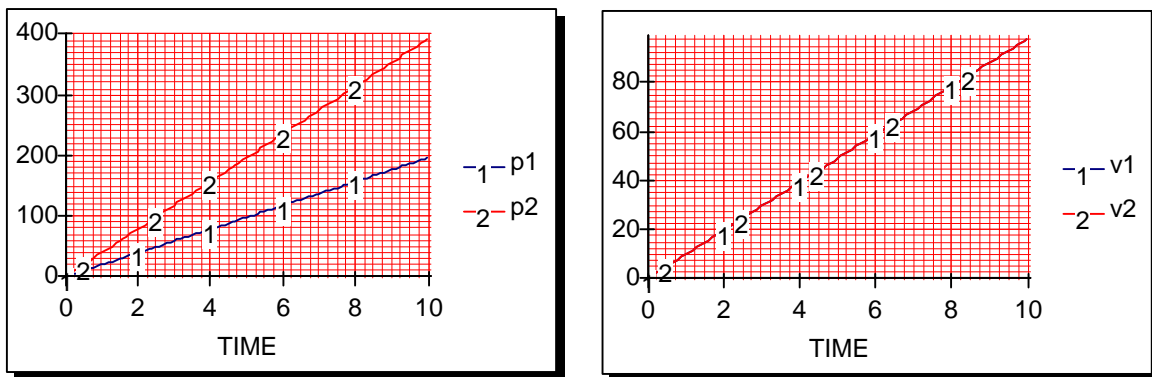


Figure 5: p - t -diagram (left) and v - t -diagram (right)

Powersim is able to display all kinds of time graphs, phase-diagrams and time tables for all quantities used in the model. In fig. 5 we see the p - t - and the v - t -diagrams for two different masses. As expected, the curves are straight lines and the v - t -diagrams are the same. We see, that the accumulation of momentum in a body during free falling leads to an increase in velocity. In other words: the increase in velocity indicates that momentum is accumulating. This is different to what is stated in EINSTEIN's Special Theory of Relativity. As we will see, momentum can be accumulated in a body without the velocity increasing.

⁷ See further information in [7] or standard books on numerical integration.

Fig. 6 shows the modified flow diagram. Energy is added as a variable. The result of the simulation is shown in fig. 7. The energy E is initialized with 0J and the energy current is given by $P = F \cdot v$. Therefore E is not the total energy of the body but only part of it. This part is traditionally called kinetic energy.

The result of the simulation is displayed in a phase-diagram window (fig.7). Here we can see how energy is dependent upon momentum. It is clear that the curve, a parabola, represents the well-known formula:

$$E_{kin} = \frac{p^2}{2m}.$$

This can be checked with the help of a table graph, generated by Powersim. (see fig. 7).

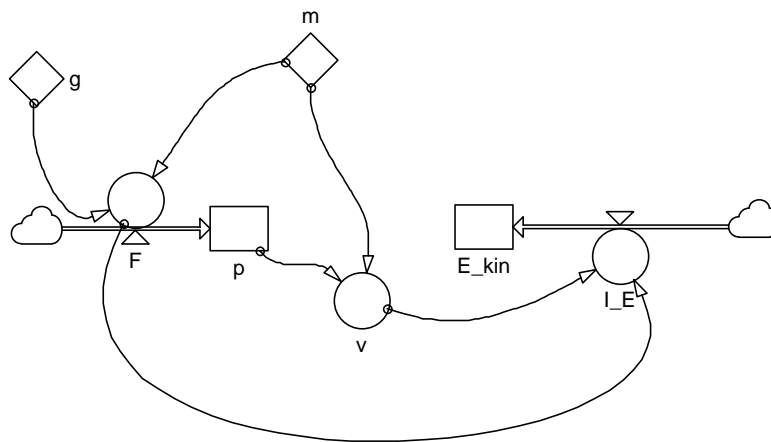


Figure 6: Flow diagram - free falling of a body on the surface of earth. Powersim does not distinguish between normal letters and capital letters, so energy current is written as I_E instead of P .

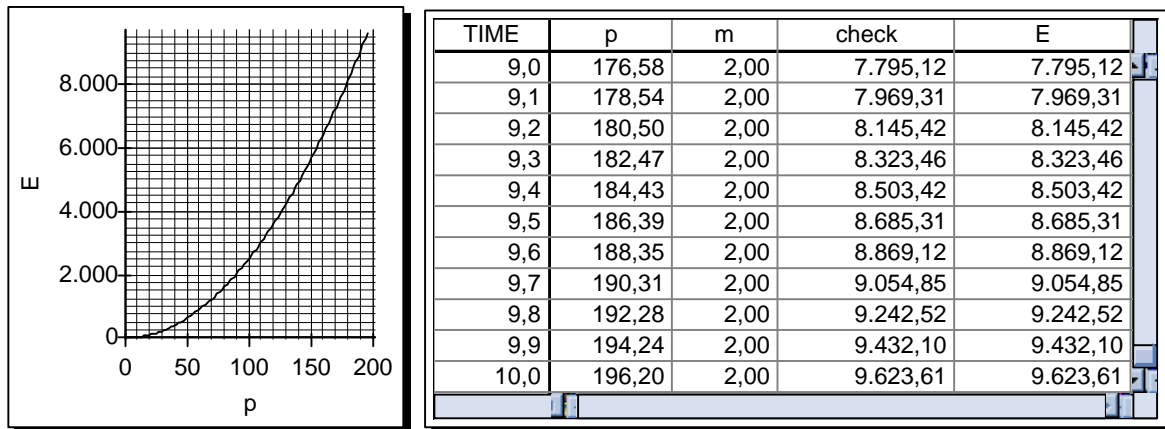


Figure 7: E - p -diagram (left). Table graph showing the last ten time steps of the iteration loop. The check column contains $p^2/(2m)$ (right).

in mass represents ‘high relativistic mechanics’. Fig. 9 (right) verifies this. While v tends towards c , the mass grows infinitely⁹ (fig. 10 left). This singularity shows that c has the same value in all reference frames. If this was not the case a body could have ‘nearly infinite’ mass and it would not be possible to accelerate this body anymore, whereas the same body could be accelerated if it was seen in another reference frame.

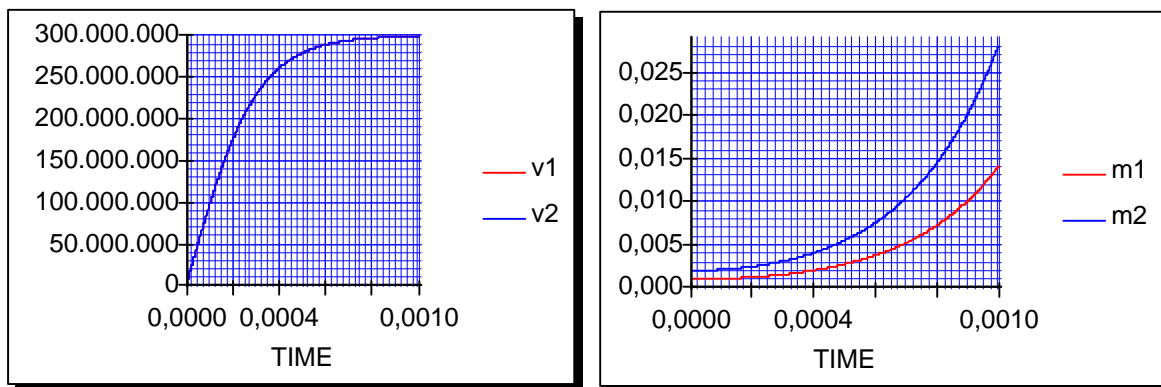


Figure 9: v - t -diagram (left) and m - t -diagrams for free falling of bodies on the surface of a neutron star.

Very interesting is the comparison of the E - p - phase-diagram shown in fig. 10 (right) and fig. 7 (left). Even when there is little momentum, the relativistic E - p -diagram can be approximated by the non-relativistic E - p -diagram, which is shifted by the rest energy (fig. 11).

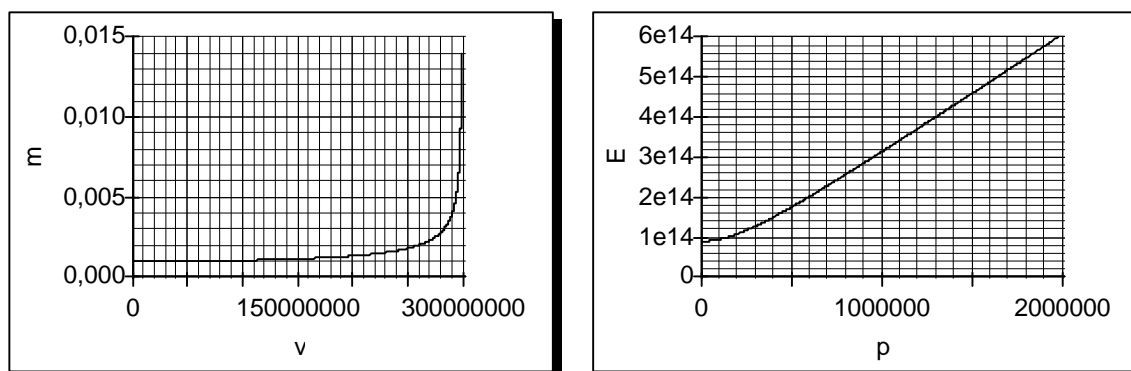


Figure 10: m - v - (left) and E - p -diagram (right)

⁹ The rate of acceleration of a body falling in a homogenous gravity field is contrasted to that of a body in a homogeneous electric field; as a result of the increase in mass while falling the momentum current into the body must increase in the same way. In a homogeneous electric field, the momentum current remains constant because the electric charge, by which the body is coupled to the electric field does not change its value while falling.

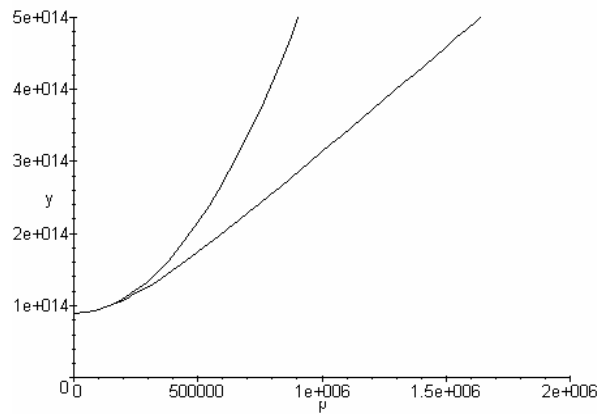


figure 11: non-relativistic and relativistic E - p -diagram

With increasing momentum all curves approach the asymptote $E = c \cdot p$ [8]. For particles which only exist at the velocity c , the formula $E = c \cdot p$ is exact. Photons are such particles. Energy and momentum of a photon are fixed by its frequency¹⁰, so the border velocity is found to be the speed of light. As c , the border velocity, is an universal constant, the speed of light must have the same value in all reference frames.

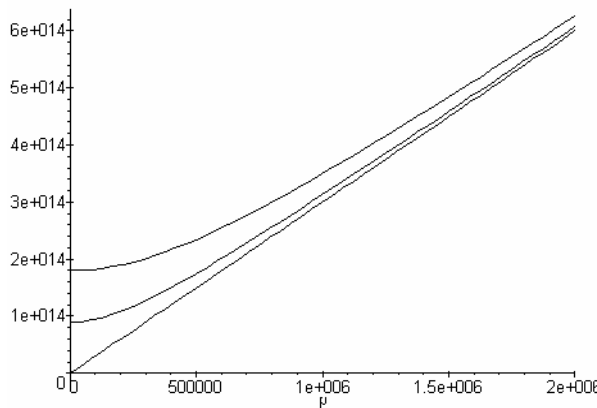


Figure 12: E - p -diagrams for the rest masses 1g and 2g. The asymptote is added

4. Proving the new way

A momentum current

$$F = I_p = \frac{dp}{dt}$$

is always coupled to an energy current according to

$$I_E = v \cdot I_p,$$

¹⁰ Let a photon fall in a gravity field, its energy and momentum will increase, this is only possible by increasing its frequency (Doppler effect)

or

$$\frac{dE}{dt} = v \cdot \frac{dp}{dt},$$

or

$$dE = v dp.$$

This is better known as the GIBBS fundamental form of a system, all of whose independent extensive variables except p are held constant. According to $E = m \cdot c^2$ and $p = m(v) \cdot v$ the GIBBS fundamental form changes to

$$d(c^2 \cdot m) = v \cdot d(m(v) \cdot v)$$

Due to the evaluation of the differentiation and separation of the variables we get

$$c^2 \cdot dm = v^2 \cdot dm + v \cdot m(v) \cdot dv$$

$$(c^2 - v^2)dm = v \cdot m(v) \cdot dv$$

$$\frac{1}{m(v)} \cdot dm = \frac{v}{c^2 - v^2} \cdot dv.$$

After integration we have

$$\begin{aligned} \int_{m(v=0)}^{m(v)} \frac{1}{m^*} dm^* &= \int_0^v \frac{v^*}{c^2 - v^{*2}} \cdot dv^* \\ &= -\frac{1}{2} \int_0^v \frac{-2v^*}{c^2 - v^{*2}} \cdot dv^* \end{aligned}$$

$$[\ln m^*]_{m(v=0)}^{m(v)} = -\frac{1}{2} \cdot [\ln(c^2 - v^{*2})]_0^v.$$

With $m(v=0) = m_0$ we get

$$\begin{aligned} \ln \frac{m(v)}{m_0} &= \\ &= -\frac{1}{2} \cdot \ln \frac{c^2 - v^2}{c^2} = \ln \sqrt{\frac{c^2}{c^2 - v^2}} = \ln \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}$$

and at least we have

$$m(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (\text{I})$$

For $v > c$ the square root in (I) is not defined. Therefore we have an important result: the physical meaning of the factor c^2 , which was originally constructed to transform kg into J, is now found to be a special velocity - the border-velocity for all momentum-energy-transports, and if $v \rightarrow c$ then $m(v) \rightarrow \infty$. Thus velocity is a singularity. This means that c has the same value in all reference frames.

With (I) we have

$$E(v) = \frac{E_0}{\sqrt{1 - \frac{v^2}{c^2}}} . \quad (\text{II})$$

We substitute for

$$\frac{v^2}{c^2} = \frac{v^2 \cdot m^2(v) \cdot c^2}{c^2 \cdot m^2(v) \cdot c^2} = \frac{p^2 \cdot c^2}{E^2(v)}$$

and get

$$E(v) = \frac{E_0}{\sqrt{1 - \frac{p^2 \cdot c^2}{E(v)^2}}}$$

and at last we have the very important equation

$$E(v) = \sqrt{E_0^2 + p^2 c^2} . \quad (\text{III})$$

For a particle which exists only at the speed of c , such a particle has no rest-energy, equation (III) is modified to $E(v) = p \cdot c$. We will get the same relation if $p^2 \cdot c^2 \gg E_0^2$.

5. Some further Consequences

According to relation $E = h \cdot f$, which shows how the energy of a photon depends on its frequency, equation (II) turns to

$$f(v) = \frac{f_0}{\sqrt{1 - \frac{v^2}{c^2}}} . \quad (\text{IV})$$

This relation means: if a person A measures the frequency of light as f_0 , another person B, to whom the first person A moves with a velocity v will measure the frequency of the 'same light' as $f(v)$ according to relation (IV). In order to measure the length of a body, person A uses

standing waves of light with the frequency f_0 . Let us assume, A counts n antinodes. Therefore he can measure the length of a body as:

$$l_0 = n \cdot \frac{l_0}{2} = \frac{n}{2} \cdot \frac{c}{f_0}.$$

Person B observes the experiment made by A. He counts the same number of antinodes as person A, but in this case, the light has the frequency $f(v)$. Thus for B the length of the body is:

$$l(v) = n \cdot \frac{l(v)}{2} = \frac{n}{2} \cdot \frac{c}{f(v)}.$$

With

$$\frac{l(v)}{l_0} = \frac{f_0}{f(v)} \text{ and equation (IV) we get the relation}$$

$$l(v) = l_0 \sqrt{1 - \frac{v^2}{c^2}} \quad (\text{V})$$

the FITZGERALD-LORENTZ contraction, which states that a body moving with a velocity v relative to an observer is contracted.

Let us assume, person A moves again with the velocity v relative to person B. B has marked in his reference frame a distance l_0 in the same direction of v . In cause of the velocity, A sees the marked distance as $l(v) = l_0 \sqrt{1 - \frac{v^2}{c^2}}$. Person A measures his velocity relative to B as

$$v = \frac{l(v)}{t_0}. \quad (\text{VI})$$

The same velocity is measured by B using the distance l_0 . Therefore he must use a time $t(v)$ to get

$$v = \frac{l_0}{t(v)}. \quad (\text{VII})$$

Using the equations (VI) and (VII) we get the equation

$$t(v) = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (\text{VIII})$$

which describes the slowing down of a moving clock or time dilatation.

6 References

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